

Symmetries and fermion masses

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Abstract

We discuss whether quark, charged lepton and neutrino masses and mixing angles may be related by an extended flavour and family symmetry group. We show that current measurements of all fermion masses and mixing angles are consistent with a combination of an underlying $SU(3)$ family symmetry together with a GUT symmetry such as $SO(10)$. In this the near bi-maximal mixing observed in the neutrino sector is directly related to the small mixing observed in the quark sector, the difference between quark and lepton mixing angles being due to the see-saw mechanism. Using this connection we make a detailed prediction for the lepton mixing angles determining neutrino oscillation phenomena.

1 Introduction

While the measurement of quark and lepton masses and mixing angles continues to improve, their theoretical understanding remains elusive. In the Standard Model the quark masses and mixing angles and charged lepton masses are determined by the Yukawa couplings, parameters which must be specified when defining the Standard Model. The renormalisable couplings of the Standard Model do not allow neutrino masses although they can be introduced through the addition of higher dimensional operators, presumably originating in physics Beyond the Standard Model.

There are a few promising suggestions for structure Beyond the Standard Model which address the fermion mass problem. For simple symmetry breaking schemes Grand Unification can relate quark and lepton masses. The most promising of such relations is the equality $m_b = m_\tau$, a result which applies at the GUT scale. Radiative corrections are dominated by QCD interactions which increase the bottom quark mass at low energy scales, giving a prediction for the bottom quark mass in good agreement with experiment. A fairly straightforward generalisation of such GUT relations, which invokes a new family symmetry, also provides good relations between the down quarks and charged leptons of the first two generations [?].

However the recent measurement of neutrino masses and mixing angles has shed doubt on the validity of simple relations between quark and lepton masses and mixing angles. The most obvious difference is the fact that neutrinos have masses much smaller than those of the quarks or charged leptons. In the context of Grand Unification the “see-saw” mechanism [?, ?, ?] provides an elegant mechanism to explain this difference. In this scheme the Standard Model spectrum is extended to include the right-handed ($SU(2)$ singlet) partners of the SM neutrinos. As these states can acquire $SU(3) \times SU(2) \times U(1)$ invariant Majorana masses, naturalness implies they should acquire mass at the scale (the GUT scale?) at which the theory Beyond the Standard

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Model breaks to the Standard Model. In addition one expects Dirac masses between the left- and right- handed neutrino states. These are generated by electroweak symmetry breaking in a similar way to that of the charged leptons and quarks and so it is reasonable to assume they will be of comparable magnitude. Diagnosing the matrix of Dirac and Majorana masses gives the mass matrix for the three light states, dominantly composed of the doublet neutrinos, of the form

$$M_{eff} = M_D^\nu \cdot M_\nu^{-1} \cdot M_D^{\nu T} \quad (1)$$

Here M_ν is the 3×3 matrix of Majorana masses for the three generations of right-handed neutrinos and M_D^ν is the 3×3 matrix of Dirac neutrino masses. Given that the natural size of M_D^ν is much less than that of M_ν , it is clear this offers an elegant explanation for the smallness of neutrino masses.

There remains the question why lepton mixing angles should be so different from quark mixing angles. This is the question we address here in the light of new information about the quark mass matrix that has recently been obtained from an analysis of the wealth of b-quark data and the information from neutrino oscillations. Our approach is to look for the largest symmetry consistent with the fermion masses concentrating on the possibility that quark and lepton masses are related. We show that the data in the quark sector data is consistent with an underlying (spontaneously broken) non-Abelian family symmetry together with an extension of the Standard Model flavour symmetry to include an $SU(2)_R$ flavour symmetry acting on the right-handed components. The charged lepton masses and neutrino masses and mixings are consistent with this symmetry provided the flavour symmetry is extended, for example to $SO(10)$, to generate GUT relations between lepton and quarks. This structure can lead quite naturally to large lepton mixing angles. Indeed, due to the see-saw mechanism, lepton mixing angles are large because quark mixing angles are small [?, ?]!

2 The quark mass matrix

Our starting point is the form of the quark mass matrix. Since experiment is only able to determine the mass eigenvalues and the CKM mixing matrix, it is not possible unambiguously to determine the quark mass matrix because the full form of the left- handed and right-handed rotation matrices needed to diagonalise the quark masses is not known. However the very reasonable assumption that the smallness of the mixing angles is due to the smallness of the mixing angles in the up- and down- sectors separately, allows one to determine the mass matrix elements on and above the diagonal to good precision for the down quarks and to lesser precision for the up quarks. In reference [?], under this assumption, the quark mass matrices were determined using the most recent experimental data. This gave the form

$$\frac{M^u}{m_t} = \begin{pmatrix} 0 & b'\epsilon^3 & c'\epsilon^3 \\ b'\epsilon^3 & \epsilon^2 & a'\epsilon^2 \\ ? & ? & 1 \end{pmatrix} \quad (2)$$

and

$$\frac{M^d}{m_b} = \begin{pmatrix} 0 & b\bar{\epsilon}^3 & c\bar{\epsilon}^3 \\ b\bar{\epsilon}^3 & \bar{\epsilon}^2 & a\bar{\epsilon}^2 \\ ? & ? & 1 \end{pmatrix} \quad (3)$$

In this the parameters of the up quark mass matrix are given by $\epsilon = 0.05$, $b' \simeq e^{i\phi}$, where in the phase convention used here $\phi \approx 90^\circ$ is the ‘standard’ (i.e. PDG convention) CP violating phase δ [?] while a' and c' are very weakly constrained. The parameters of the down quark mass matrix are much better determined with

$$\begin{aligned}\bar{\epsilon} &= 0.15 \pm 0.01 & b &= 1.5 \pm 0.1 & a &= 1.31 \pm 0.14 \\ |c| &= 0.4 \pm 0.05 & \psi &\equiv \text{Arg}(c) = -24^\circ \pm 3^\circ & \text{or} \\ |c| &= 1.28 \pm 0.05 & \psi &= 60^\circ \pm 5^\circ\end{aligned}\tag{4}$$

In both M^u and M^d the entries marked with a question mark are only weakly constrained. In what follows we shall consider the case that the mass matrices are symmetric or antisymmetric or some combination of the two so that the elements below the diagonal are also determined. This is in the spirit of looking for the maximal symmetry consistent with the masses and is motivated by certain GUTs, notably $SO(10)$. There is one strong piece of experimental evidence for such a symmetric structure, namely the success of the Gatto, Sartori, Tonin relation [?] between the Cabibbo angle and the quark masses of the first two generations

$$V_{us} = \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} e^{i\sigma}\tag{5}$$

where σ is the CP violating phase entering the Jarlskog invariant [?, ?]. This relation only applies if the (1, 2) and (2, 1) matrix elements are equal in magnitude. The solution of eqs(2) and (3) has this structure. Of course there is no direct indication that the same reflection symmetry applies to the remaining matrix elements but it is the most natural generalisation of this result. In any case, as stressed above, we think it of interest to ask whether the most symmetric form for the quark mass matrices can be simply related to viable forms for the lepton mass matrices.

3 Quark family and flavour symmetries

In this Section we will discuss whether the quark mass matrices are consistent with a larger set of symmetries than those of the Standard Model³. These may be flavour symmetries acting in the same way on each of the three families; for example simple Grand Unified theories are flavour symmetries. Alternatively they may also be family symmetries distinguishing between families. In our opinion the structure of the quark mass matrices strongly suggests an underlying (spontaneously) broken family symmetry. In such a scheme, in leading order, only the (3, 3) elements are allowed. In terms of the parameter, ϵ , characterising the symmetry breaking the remaining matrix elements are filled in at some order ϵ^n which is also determined by the symmetry, thus generating the hierarchical structure of fermion masses.

The candidate family symmetry group, namely those symmetries which commutes with the gauge interactions of the Standard Model, is quite large. For the quarks it is $U(3)^3$, corresponding to a separate $U(3)$ factor for the left-handed doublets and each of the right-handed $SU(2)$ singlet fields respectively. To generate quark mass matrices with a left- right- symmetry we restrict the

³Here we shall only consider supersymmetric theories.

choice to one with the same transformation for the left-handed and the charge conjugate right-handed quarks, this symmetry is reduced to a diagonal $U(3)$ or one of its subgroups⁴.

Abelian family symmetries have been widely explored. They are capable of explaining the hierarchical structure of the quark masses and generate quite acceptable forms for the quark matrices [?, ?]. The main shortcoming of these schemes is that they do not give a reason for the smallness of V_{cb} . This follows because if one assigns Abelian charges to generate the mass hierarchy between m_s and m_b , it leads to the prediction for $V_{cb} \simeq \sqrt{m_s/m_b}$ which is too large [?, ?]. Although this relation is quite sensitive to corrections of $O(1)$ due to unknown Yukawa couplings, it is an unsatisfactory feature of an Abelian symmetry. To obtain the correct form for V_{cb} it is necessary for the ratio of the (2, 2) and (2, 3) matrix elements to be close to 1; a ratio of 1 implies $V_{cb} \simeq m_s/m_b$, in good agreement with experiment. In contrast to the Abelian case a non-Abelian family symmetry can relate the magnitude of the couplings in different positions in the mass matrix and thus offers the possibility of explaining this near equality.

3.1 Non-Abelian family symmetry

The largest non-Abelian symmetry which treats the left- and charge conjugate right-handed components in the same way is $SU(3)$. In [?] a specific $SU(3)$ theory has been constructed which relates the (2, 2) and (2, 3) matrix elements. In it the left-handed quark doublets and the up and down charge conjugate quarks are assigned to $SU(3)$ triplets, ψ_i , u_i^c and d_i^c respectively. The critical part of the theory lies in the pattern of symmetry breaking which leads to the desired mass matrix. This is achieved through the antitriplet scalar fields ϕ_{23}^i , ϕ_3^i and triplet fields $\bar{\phi}_{23,i}$, $\bar{\phi}_{3,i}$ which acquire vacuum expectation values (VEVs) $\phi_3^T = \begin{pmatrix} 0 & 0 & M \end{pmatrix}$, $\phi_{23}^T = \begin{pmatrix} 0 & a_2 & a_2 \end{pmatrix}$, $\bar{\phi}_3 = \begin{pmatrix} 0 & 0 & M \end{pmatrix}$ and $\bar{\phi}_{23} = \begin{pmatrix} 0 & a_2 & -a_2 \end{pmatrix}$. In [?] it was shown how such vacuum alignment can be achieved in a supersymmetric theory, with $M > a_2$.

With this pattern of symmetry breaking it is easy to generate the required form of the mass matrices via the superpotential

$$P_{Yukawa} = (\psi_i^c \phi_3^i \phi_3^j \psi_j + \psi_i^c \phi_{23}^i \phi_{23}^j \psi_j) H_\alpha / M^2 + \epsilon^2 \epsilon^{ijk} \psi_i^c \bar{\phi}_{23,j}^i \psi_k^c H_\alpha / M + \epsilon^6 \left(\epsilon^{ijk} \psi_i^c \bar{\phi}_{23,j}^i \bar{\phi}_{3,k}^c \right) \left(\epsilon^{ijk} \psi_i \bar{\phi}_{23,j}^i \bar{\phi}_{3,k}^c \right) H_\alpha / M^4 \quad (6)$$

where $H_{1,2}$ are the Higgs doublets of the MSSM, which are $SU(3)$ singlets. If $\psi^c = u_i^c$, d_i^c then $\alpha = 2, 1$ respectively, giving rise to the up and down quark masses in a supersymmetric theory. In eq(6) we have suppressed the couplings of $O(1)$ associated with each operator. The first two operators shown are the leading (dimension 6) ones consistent with a simple family symmetry⁵[?]. The third and fourth operators are only generated via higher dimension operators involving additional powers of fields ϕ_{23} , ϕ_3 . Replacing these fields by their vevs leads to the suppression factors ϵ^2 and ϵ^6 shown, where $\epsilon = a_2/M$. This gives rise to the mass matrices of

⁴It is possible there is an even larger symmetry with separate gauge group factors operating on left-and right-handed components and related by a discrete reflection symmetry. If one requires the left- right- symmetry is preserved on spontaneous symmetry breaking, the phenomenological implications of this scheme are the same as the case considered here.

⁵In a specific realisation of the scheme it is necessary to introduce further Abelian family symmetries to ensure this is the most general form. A specific realisation is given in reference [?].

the form

$$\frac{M}{M_{3,3}} = \begin{pmatrix} \lambda'\epsilon^8 & \lambda\epsilon^3 & \lambda\epsilon^3 \\ -\lambda\epsilon^3 & \lambda''\epsilon^2 & \lambda''\epsilon^2 \\ -\lambda\epsilon^3 & \lambda''\epsilon^2 & 1 + \lambda''\epsilon^2 \end{pmatrix} \quad (7)$$

where the expansion parameter ϵ may differ for the up and down sectors and we have explicitly included the $O(1)$ coefficients λ , λ' , λ'' that arise due to the $O(1)$ couplings in eq(6) which are not related by the non-Abelian symmetry. The effect of the $SU(3)$ family symmetry may be readily seen relating the (1, 2) and (1, 3) matrix elements as well as the (2, 2) and (2, 3) matrix elements. Of course there are corrections to these equalities coming at higher order in ϵ from operators of higher dimension. We shall consider their effect in detail later.

The form of eq(7) gives good agreement with the measured quark mass matrices and can also be extended to give a viable description of charged lepton and neutrino masses and mixing angles [?]. Here we consider whether the $O(1)$ coefficients can be related by further symmetries and whether it is possible to determine the lepton masses and mixing angles in terms of the quark masses and mixing angles.

3.2 $SU(2)_R$ and Grand Unification

In implementing the $SU(3)$ family symmetry the family structure has been strongly constrained to generate left-right symmetry by assigning the left- and the charge conjugate of the right-handed components of a given quark to the same representation. Since, in addition, the family symmetry must respect the $SU(2)_L$ gauge symmetry the up and down left-handed components of a given quark family doublet must also carry the same family charge. As a result of these two constraints one immediately sees that left- and charge conjugate right- handed components of the up and down quark members of a given family have the same family charge. This means that the up and down quark mass matrices have the same *form* as in eqs(2) and (3), although the expansion parameters may be different and the operator coefficients of $O(1)$ may differ.

If $SU(2)_R$ is also an exact symmetry of the theory even the expansion parameters and operator coefficients are equal and the mass matrices are identical. This is clearly not acceptable and, if there is an underlying $SU(2)_R$ symmetry, it must be spontaneously broken so that the equality of the mass matrices will be lost through soft symmetry breaking terms. These enter through the expansion parameter ϵ which is determined by θ/M where θ is the field spontaneously breaking the symmetry and M is the messenger mass of the state responsible for communicating the symmetry breaking and generating the higher dimension operators. Due to $SU(2)_R$ breaking the messenger mass may be different for the up and down quark sectors and hence the expansion parameters may differ. It is also possible that the family symmetry breaking field, θ , is not a singlet under $SU(2)_R$ and its vev breaks $SU(2)_R$, again leading to a different expansion parameter for the up and the down sectors. However there will still be some measurable effects of an underlying structure $SU(2)_R$ in the quark masses because the Yukawa couplings of the field θ to the quarks and the messenger sector, which are responsible for the $O(1)$ coefficients, will respect the symmetry. How large are the corrections to this result? They occur in radiative order but, in a supersymmetric theory, supersymmetry guarantees that superpotential is not renormalised. In this case the dominant radiative corrections to the Yukawa couplings come from the D-terms controlling wavefunction renormalisation. Thus we may expect that fields falling into $SU(2)_R$

doublets will have normalisation which differ only in higher order, $O(|\theta|^2/M^2)$. The resulting corrections to the $O(1)$ coefficients will also be at this order.

What are the phenomenological implications of an underlying $SU(2)_R$? If only the leading operator contributions to eq(7) are included the coefficients a , b and c in eq(3) will equal a' , b' and c' in eq(2) respectively⁶. We wish to explore whether such a broken symmetry expansion is consistent with the quark and lepton masses and mixings. Unfortunately the tests of the $SU(2)_R$ relations in the quark sector are somewhat imprecise at present because the contribution to the mixing angles from the up sector is small and sensitive to small changes in the down sector. As a result a' and c' are only poorly determined. Given that in eqs(2) and (3) the $(2, 2)$ matrix elements have been used to define the expansion parameters, the remaining test is to compare b and b' . It is easy to eliminate the dependence on the expansion parameters by combining $(1, 1)$, $(2, 2)$ and $(1, 3)$ elements. Using eqs(2) and (3) this gives

$$\frac{b'}{b} = \frac{m_s}{m_c} \sqrt{\frac{m_u m_t}{m_d m_b}} \quad (8)$$

where this relation applies at the unification scale where $SU(2)_R$ is a good symmetry. This gives

$$\frac{b'}{b} = 0.5 \pm 0.3 \quad (9)$$

Given that the expansion parameter in the down quark sector is quite large, $\bar{\epsilon} = 0.15$, and that higher order corrections may be significant, the result of eq(9) may be consistent with an underlying $SU(2)_R$ relation prediction between b' and b , with a significant correction at $O(\theta/M)$ coming from such higher dimension operators. We shall return to a discussion of this possibility in Section 4.

Of course it is likely that $SU(2)_R$ is part of an underlying GUT such as $SO(10)$. In this case the question whether $SU(2)_R$ is an approximate symmetry of the couplings depends on the pattern of symmetry breaking. We shall also discuss a second possibility for reconciling the GUT prediction with eq(9) in which the couplings themselves feel $SU(2)_R$ breaking.

3.3 GUTs and charged lepton masses

We have seen that the quark mass matrices are consistent with the choice that all the states of the (left-handed) components of a given family have the same transformation properties under the family symmetry. This is suggestive of a larger underlying GUT symmetry. The GUT $SO(10)$ is particularly promising as all the (left-handed) states of a family, plus the charge conjugate of the right handed neutrino, fit into a single 16 dimensional representation. If $SO(10)$ is an underlying symmetry of the theory any family symmetry must commute with it implying all the charges of a given family must be the same, consistent with the form of the mass matrix discussed above. An associated advantage of such a family symmetry is that the mixed anomalies of the Standard Model gauge group with the family symmetry will automatically cancel because of the structure of the underlying $SO(10)$. It also contains $SU(2)_R$ and can relate the up to the down sectors as discussed above. In addition a GUT symmetry can relate quark and lepton mass matrices and this is the issue we wish to study here.

⁶In general we expect higher order operators contributing to a given matrix element at $O(\theta/M)$ or above relative to the leading term.

We first consider the charged leptons. Given the same family properties the form of the charged lepton mass matrix will be the same as that of the down quark matrices in eq(3) up to coefficients of $O(1)$. If there is an underlying GUT the coefficients too may be related. As mentioned above the relation $m_b = m_\tau$ at the unification scale is in good agreement with the measured masses. Such an equality applies in $SU(5)$ if the Higgs responsible for the third generation masses transforms as a $\bar{5}$ of $SU(5)$. In $SO(10)$ equality applies if the Higgs belongs to a 10 representation, this also gives equality between the top quark and the third generation Dirac neutrino mass, something we explore below.

What about the two lighter generations? In this case we must address the question whether the expansion parameters are related. From a phenomenological point of view, note that, after taking radiative corrections into account, the relation $Det[M^d] = Det[M^l]$ at the unification scale is also in good agreement with the experimental measurements. From eq(7) we see $Det[M]/M_{33} = \lambda^2 \epsilon^6$ and so equality requires that the $(1, 2)$ matrix element of magnitude $\lambda \epsilon^3$ be the same for the down quarks and the leptons. This is consistent with an underlying broken $SU(2)_R$ symmetry because the down quarks and leptons are both $T_{R,3} = -1/2$ states and both can acquire their mass from the same Higgs doublet, H_2 , in a supersymmetric theory. Thus the strong $SU(2)_R$ breaking, needed to split the $T_{R,3} = \pm 1/2$ states, is consistent with this equality.

3.3.1 Symmetry breaking expansion parameters

Of course equality of the down quark and charged lepton matrix elements in the $(1, 2)$ position requires that the expansion parameters be the same in the two sectors. This is consistent with an underlying broken $SU(2)_R$ symmetry because the down quarks and leptons are both $T_{R,3} = -1/2$ states and both can acquire their mass from the same Higgs doublet, H_2 , in a supersymmetric theory. Thus the strong $SU(2)_R$ breaking, needed to split the $T_{R,3} = \pm 1/2$ states, does not differentiate between the down quarks and charged leptons. Clearly if the dominant messenger sector for family symmetry breaking is in the Higgs sector the expansion parameter in these two sectors will be equal. Similarly, in this case, the expansion parameter in the up quark and neutrino sectors will be the same, since both acquire Dirac masses from the same Higgs doublet, H_1 . A similar conclusion applies if the family symmetry breaking field, θ , is not a singlet under $SU(2)_R$ and its vev breaks $SU(2)_R$, again leading to a different expansion parameter for the up and the down sectors.

The other possibility is that there are significant contributions from messengers carrying quark and lepton quantum numbers. For the messengers carrying left-handed generation quantum numbers, $SU(2)_L$ requires the messenger masses should be equal so the only way that these terms could be consistent with the up and down masses is if the family symmetry breaking field, θ , is not a singlet under $SU(2)_R$ and its vev breaks $SU(2)_R$. Finally it is possible that the messengers carry the quantum numbers of the left-handed charge conjugate generations. In this case the up and down sectors could, through $SU(2)_R$ breaking, have a different masses leading to different expansion parameters. If the underlying symmetry breaking pattern is

$$SO(10) \rightarrow SU(4) \times SU(2)_L \times SU(2)_R \rightarrow SU(4) \times SU(2)_L \times U(1) \quad (10)$$

the down quarks and charged leptons will have the same expansion parameter and so will the up quarks and neutrinos.

In what follows we will explore the most predictive possibility, namely that there is an expansion parameter $\bar{\epsilon}$ which applies to the down quarks and charged leptons, and an expansion parameter ϵ which applies to the up quarks and neutrinos.

3.3.2 Quark/lepton couplings

If the expansion parameter is indeed the same for the down quarks and leptons the quantities $Det[M^d]$ and $Det[M^l]$ occur at the same order as is required phenomenologically. Exact equality requires also the equality of the coefficients determining the $(1, 2)$ matrix elements. Just as for the $(3, 3)$ elements such equality will follow if the Higgs responsible for this element transforms as a $\bar{5}$ of $SU(5)$ or to a 10 dimensional representation of $SO(10)$. However it is not possible to have *identical* charged lepton and down quark mass matrices because, after taking account of the radiative correction on going from high to low scale, giving approximately a factor of 3 increase in the quark masses, the relations $m_s = m_\mu$ and $m_d = m_e$ are in gross disagreement with experiment. As pointed out by Georgi and Jarlskog [?] this discrepancy is readily explained if there is an underlying GUT through the appearance of Clebsch Gordon factors in the matrix element coefficients. In particular if the Higgs responsible for the $(2, 2)$ matrix element should belong to a $\bar{45}$ of $SU(5)$ (or 126 of $SO(10)$) the lepton coupling is a factor -3 times the down quark coupling. In this case, taking account of the equality of the determinants, the relations for the light generations are modified to give $m_s = m_\mu/3$ and $m_d = 3m_e$. Including the radiative corrections needed to determine the masses at laboratory scales, these relations are in excellent agreement with the measured masses. If there is also an underlying non-Abelian family symmetry relating the matrix elements as in eq(3.1), the $(2, 2)$ and $(2, 3)$ matrix elements are related and so it is necessary that the Higgs field responsible for the $(2, 3)$ matrix element be also due to a Higgs the $\bar{45}$ of $SU(5)$. With this the resulting form of the lepton mass matrix coming from eq(3) is given by

$$M^l \simeq \begin{pmatrix} \bar{\epsilon}^8 & b\bar{\epsilon}^3 & c\bar{\epsilon}^3 \\ b\bar{\epsilon}^3 & -3\bar{\epsilon}^2 & -3a\bar{\epsilon}^2 \\ c\bar{\epsilon}^3 & -3a\bar{\epsilon}^2 & 1 \end{pmatrix} \quad (11)$$

Note that in order to realise this scheme there must be a very particular origin to the effective Lagrangian given in eq(6). The effective operators may be generated through quark or Higgs mixing with states carrying different family symmetry quantum numbers. In order to generate the factor 3 it is necessary that the operators responsible for these terms be dominantly given by Higgs mixing of the $\bar{5}$ Higgs responsible for the $(3, 3)$ element with the $\bar{45}$ generating the factor -3 enhanced terms. Note that it is not necessary for the $\bar{45}$ to be an elementary Higgs field, it can arise as an effective Higgs, for example from the coupling of a $\bar{5}$ and 24 (in $SO(10)$ the 126 can be generated by a 10 and 45) as in Figure 1[?]. In these graphs $\bar{\chi}$ and χ ($\bar{\chi}^c$ and χ^c) are a vectorlike pair of chiral supermultiplets with mass M (M'), where χ (χ^c) has the same SM quantum numbers as the left-handed generations ψ (left-handed antigenerations ψ^c).

Since the adjoint fields are already needed to break $SU(5)$ (or $SO(10)$) this represents a simplification of the Higgs sector. Moreover we see that such graphs can also generate a coupling to an effective 120 dimensional representation of $SO(10)$. This coupling would be antisymmetric in family space if the 120 were fundamental but need not be so in the case they are given by Figure 1 because the intermediate messenger masses associated with the graphs coupling to H_{10}

and Σ_{45} are not necessarily the same as those coupling to Σ_{45} and H_{10} . Writing the vacuum expectation value

$$\langle \Sigma \rangle = B - L + \kappa T_{R,3}, \quad (12)$$

the relative contribution of these graphs to the down quarks and leptons is respectively 1/3 and

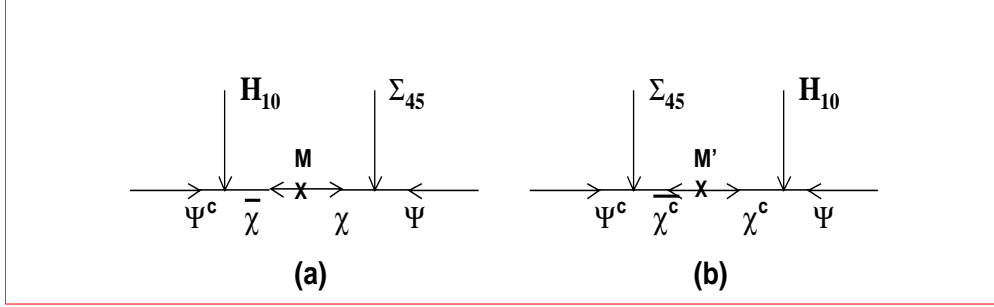


Figure 1: Froggatt Nielsen supergraphs generating fermion masses.

-1 for Figure 1a and $(1/3 - \kappa/2)$ and $(-1 - \kappa/2)$ for Figure 1b. Consider the case the second graph has lighter messengers and dominates. For $\kappa = 0$ we obtain the form of eq(11). However for case $\kappa = 2$ the contribution to the leptons is +3 times that of the quarks so we obtain eq(11) with +3 rather than -3. Since the lepton mass eigenvalues are insensitive to this sign we obtain identical masses for either form. This case corresponds to the coupling of an effective 120 dimensional representation and has different implications for the structure of the up quarks and neutrinos that we explore below.

The mass matrix of eq(11) gives excellent relation between the down quarks and charged lepton masses of all three generations. Due to the approximate texture zero in the (1, 1) position it also implies a contribution to the (1, 2) element of the matrix V^l diagonalising the charged lepton mass matrix, given approximately by

$$V_{12}^l \simeq \sqrt{\frac{m_e}{m_\mu}}. \quad (13)$$

In turn this gives a contribution to the (1, 2) element of the MNS mixing matrix, $V^{MNS} = V^{l\dagger} V^\nu$ [?]. As we shall discuss this is significant in solar neutrino oscillations if the (now disfavoured) SMA solution applies; it is also important in determining V_{13}^{MNS} . On the other hand, due to the smallness of the (1, 3) matrix element, the contribution of the (2, 3) element of V^l to the MNS mixing matrix is small

$$V_{23}^l \simeq \frac{m_\mu}{m_\tau}. \quad (14)$$

This is the analogue to the relation in the down quark sector $V_{cb} \simeq m_s/m_b$, which is in good agreement with the small value observed. However it is too small to explain the large mixing angle observed in atmospheric neutrino oscillation. This discrepancy is at the heart of the apparent conflict between quark and lepton masses and we turn now to the discussion how this conflict may be resolved.

4 Neutrino masses.

4.1 Dirac mass

Following the discussion in Section 3 we immediately see that the $SU(3)$ family symmetry properties of the left- and right- handed neutrinos must be the same as those of the charged leptons. As a result the form of the neutrino Dirac mass matrix between the doublet neutrinos and the singlet (right-handed) neutrinos should be the same as eq(11), although the expansion parameter may be different. Given the success of GUT relations in the charged mass matrix it is obviously reasonable to explore the possibility that neutrino masses are similarly related to quark masses. As discussed above the expansion parameter for neutrino masses may be equal to that for up quark masses. This is consistent with an underlying $SU(2)_R$ (contained in $SO(10)$) because the up quarks and right-handed neutrinos are both $T_{3,R} = 1/2$ states. In this case the resulting Dirac neutrino mass matrix is of the form

$$M_D^\nu/M_{D,33}^\nu = \begin{pmatrix} O(\epsilon^8) & b'_\nu \epsilon^3 & c'_\nu \epsilon^3 \\ b'_\nu \epsilon^3 & d'_\nu \epsilon^2 & a'_\nu \epsilon^2 \\ c'_\nu \epsilon^3 & a'_\nu \epsilon^2 & 1 \end{pmatrix} \quad (15)$$

The underlying GUT symmetry relates the renormalisable couplings of the messenger states. For the case the Higgs responsible for the $(3,3)$, $(1,2)$ and $(2,1)$ matrix elements transforms as the 10 of $SO(10)$ and the Higgs responsible for the $(2,2)$, $(2,3)$ and $(3,2)$ matrix elements transforms as the 126 of $SO(10)$ we have $a'_\nu = -3a$, $d'_\nu = -3$, $b'_\nu = b$, and $c'_\nu = c$. In the case just discussed where the Higgs responsible for the $(2,2)$, $(2,3)$ and $(3,2)$ matrix elements transform as the 120 of $SO(10)$ (with the second graph of Figure 1 dominating) we have $a'_\nu = -\frac{(6-3\kappa)}{(2+3\kappa)}a$, $d'_\nu = -\frac{(6-3\kappa)}{(2+3\kappa)}$, $b'_\nu = b$, and $c'_\nu = c$

4.2 Majorana mass

Of course it is necessary to determine the Majorana mass matrix before one can determine the effective neutrino mass matrix via the see-saw formula of eq(1). Although the family symmetry properties of the right-handed neutrinos are related to those of the charged leptons it is not possible to use this information unambiguously to determine the structure of the Majorana mass matrix. In particular it may not have the same form as is found for the Dirac matrix. The reason is twofold. Firstly the Majorana masses are generated via a new $\Delta L = 2$ lepton number violating Higgs sector and it is necessary to specify the family symmetry representation content of this sector before the Majorana mass structure is fixed. Secondly the Majorana mass matrix involves the coupling of identical fermions and so antisymmetric terms allowed in the Dirac mass matrix will not arise in the Majorana matrix. Despite this we can make some general statements about the structure. In particular we expect an hierarchical structure for the Majorana mass matrix because the underlying family symmetry ($SU(3)$), is the same as applies to the Dirac matrix which leads to a structure ordered by a new expansion parameter ε_M . If a single $\Delta L = 2$ (effective) symmetry breaking field, Φ with definite $SU(3)$ family transformation properties, dominates there will no possibility of degeneracy between matrix elements. For a large part of the parameter space, the form of the Majorana mass matrix is in practice determined. In the limit that $\varepsilon_M \ll \epsilon$ (which is the case if the messenger sector in the Φ sector is heavier than in

the electroweak breaking sector) the mixing in the neutrino sector is dominated by the mixing coming from the Dirac mass. If Φ couples dominantly to the (3,3) position the resulting mass matrix can be diagonalised by small rotations of $O(\varepsilon_M)$ giving

$$M_\nu = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} \quad (16)$$

It is important to note that this mass eigenstate basis is likely to be very close to that used for the Dirac matrix, eq(15), the mass eigenstates will be the family symmetry eigenstates up to corrections of $O(\varepsilon_M)$.

This is the general form we shall use in our phenomenological analysis. It corresponds to one of the two most reasonable possibilities. By “reasonable”, we mean that large mixing should not be due to a detailed correlation between the Dirac and Majorana mass matrix elements because this would require a correlation between the $\Delta L = 2$ and $\Delta L = 0$ symmetry breaking sectors and there is no obvious symmetry that could achieve this. The other reasonable possibility is that the mixing is dominated by the mixing in the Majorana mass matrix (the case $\varepsilon_M \gg \epsilon$). This is less predictive (and hence less interesting) because, in this case, we cannot relate neutrino mixing angles with quark mixing angles. For this reason we concentrate here on the first possibility.

5 The light neutrino mass matrix

We are now able to determine the masses and mixing angles of the light neutrinos using eqs(15,16) in eq(1). The effective Lagrangian associated with the see-saw mass is of the form

$$\mathcal{L} = \frac{\epsilon^6}{m_1} (b'_\nu \nu_\mu + c'_\nu \nu_\tau + O(\epsilon^5) \nu_e)^2 H_1^2 + \frac{\epsilon^4}{m_2} (d'_\nu \nu_\mu + a'_\nu \nu_\tau + \epsilon b'_\nu \nu_e)^2 H_1^2 + \frac{1}{m_3} (\nu_\tau + a'_\nu \nu_\nu + \epsilon c'_\nu \nu_e)^2 H_1^2 \quad (17)$$

5.1 Near maximal mixing

It is now straightforward to determine the viable forms of the mass matrix. Consider first the heaviest neutrino which should have near maximal mixing to explain atmospheric neutrino oscillation. One sees from eq(17) that, surprisingly, this is easy to achieve.

We first consider the case $\frac{\epsilon^4}{m_2} > \frac{\epsilon^6}{m_1}, \frac{1}{m_3}$. The heaviest eigenstate is $\nu_1 \propto d'_\nu \nu_\mu + a'_\nu \nu_\tau + \epsilon b'_\nu \nu_e$. If $SU(2)_R$ is an underlying symmetry of the couplings ($\kappa = 0$ in eq(12)), $a'_\nu/d'_\nu \simeq a \simeq 1.3$. In this case ν_1 is very close to a maximal mixed state of muon and tau neutrinos. Note, as remarked in Section 3, it is the smallness of V_{cb} which requires a (2,3) entry of $O(\bar{\epsilon}^2)$ that leads to near maximal mixing in the neutrino sector[?].

It is also straightforward to compute the next lightest neutrino state. It will have mass of $O(\frac{\epsilon^6}{m_1} \text{ or } \frac{1}{m_3})$ whichever is the larger. Let us consider the first possibility first. The eigenstate ν_2 has the form

$$\nu_2 \propto (a'_\nu b'_\nu - c'_\nu d'_\nu)(d'_\nu \nu_\tau - a'_\nu \nu_\mu) + O(\epsilon) \nu_e \quad (18)$$

Except for the case the factor $(a'_\nu b'_\nu - c'_\nu d'_\nu)$ is anomalously small the second eigenstate will be predominately in the ν_μ, ν_τ direction. The same conclusion applies if the second eigenstate is

dominated by the third term of eq(17). Thus in both cases the mixing angle $V_{\nu_e\mu}$ relevant to solar neutrino oscillation will be dominated by the contribution of $O(\bar{\epsilon})$ from the charged lepton sector given in eq(13). This is of the correct magnitude to fit the SMA solution. The ratio of the two heaviest masses is given by $M_3/M_2 = O(\epsilon^2 m_2/m_1 \text{ or } \epsilon^4 m_2/m_3)$ and the Majorana masses can be adjusted to get the value needed for the SMA solution. For the case $\frac{\epsilon^6}{m_1} > \frac{\epsilon^4}{m_2}, \frac{1}{m_3}$ a similar pattern emerges except that the heaviest state is now $\nu_1 \propto b'_\nu \nu_\mu + c'_\nu \nu_\tau$. $SU(2)_R$ symmetry leads to the relation $b'_\nu/c'_\nu \simeq b/c \simeq 0.5$ and so the mixing is large but not maximal. Provided the factor $(a'_\nu b'_\nu - c'_\nu d'_\nu)$ is not anomalously small the lighter states are not strongly mixed and the dominant contribution to V_{12}^{MNS} comes from the charged lepton sector. From this we see that a large atmospheric neutrino mixing angle and the SMA solar neutrino solution is consistent with a very symmetrical structure for the mass matrices in which there is an approximate up-down $SU(2)_L \times SU(2)_R$ symmetry together with a GUT symmetry relating quarks and leptons.

This analysis has demonstrated that, through the see-saw mechanism, it is quite natural to get near maximal mixing in the neutrino channel for atmospheric neutrino mixing, the large lepton mixing being due to the small mixing in the quark sector! However inclusion of the recent SNO data now strongly disfavours the SMA solution for the solar mixing angle and favours large mixing in this case too. To address this we now turn to the question whether near bi-maximal mixing is consistent with an enlarged family and flavour symmetry.

5.2 Near bi-maximal mixing

At first sight it seems that our analysis has already ruled out the possibility of near bi-maximal mixing. The possible exception to this is if the factor $(a'_\nu b'_\nu - c'_\nu d'_\nu)$ is of $O(\epsilon)$, in which case the second state of eq(18) has a large ν_e component. There are two ways that this may happen naturally. The first is through $SU(2)_R$ breaking in the coefficients which occurs through spontaneous breaking via the vev of eq(12) for the case $\kappa \neq 0$. For the special case $\kappa = 2$, which generates good charged lepton masses, we have $a'_\nu = d'_\nu = 0$. In this case higher order terms in the expansion in terms of the symmetry breaking parameter can generate $(a'_\nu b'_\nu - c'_\nu d'_\nu)$ of $O(\epsilon)$ as required. The case $\kappa = 2$ can arise quite naturally on spontaneous symmetry breaking as it corresponds to an enhanced symmetry point at which the effective potential can easily have a minimum. The second way $(a'_\nu b'_\nu - c'_\nu d'_\nu)$ may naturally be of $O(\epsilon)$ applies even to the case $\kappa = 0$ and again relies on higher order terms in the expansion in terms of the symmetry breaking parameter.

Let us start with a discussion of the sub-leading terms. Consider the case of the $SU(3)$ family symmetry which leads to the leading order mass matrix of the form given in eq(7). Including higher order terms in the symmetry breaking expansion parameter gives a mass matrix of the form

$$M/M_{3,3} = \begin{pmatrix} \epsilon^8 & \epsilon^3(z + (x+y)\epsilon) & \epsilon^3(z + (x-y)\epsilon) \\ -\epsilon^3(z + (x+y)\epsilon) & \epsilon^2(aw + u\epsilon) & \epsilon^2(aw - u\epsilon) \\ -\epsilon^3(z + (x-y)\epsilon) & \epsilon^2(aw - u\epsilon) & 1 \end{pmatrix} \quad (19)$$

Here z , w and u are real coefficients and x and y complex coefficients of order 1. For the case of the down quarks and leptons the expansion parameter is $\epsilon = \bar{\epsilon}$ and for up quarks and neutrinos $\epsilon = \epsilon$ (cf. the discussion of Section 3.3.1). The higher order terms are necessary to fit

the data for M_d of eq(3). Following the discussion of Section 3.3.2 we assume that the coefficient a arises from an effective 120 or 126 $SO(10)$ representation in order to generate the correct lepton mass spectrum. Thus a is determined by the value of κ in eq(12) and there are two possibilities

$$\kappa = 0, a_l = -3, a_\nu = -3 \quad (20)$$

$$\kappa = 2, a_l = +3, a_\nu = 0 \quad (21)$$

We assume all other coefficients come from a 10 of $SO(10)$ so they are the same as for the quarks.

We may determine the neutrino effective mass matrix using the see-saw formula, eq(1). We focus on the case that the dominant contribution to the heaviest of the light neutrino states is due to the exchange of the right handed neutrino with mass m_1 which couples to the first row of the Dirac mass matrix (19). As a result it is given by

$$\nu_a = \frac{(z + x\epsilon)(\nu_\mu + \nu_\tau) + y\epsilon(\nu_\mu - \nu_\tau)}{\sqrt{(z + (x + y)\epsilon)^2 + (z + (x - y)\epsilon)^2}} \quad (22)$$

The state orthogonal to it is

$$\nu_b = \frac{(z + x\epsilon)(\nu_\mu - \nu_\tau) - y\epsilon(\nu_\mu + \nu_\tau)}{\sqrt{(z + (x + y)\epsilon)^2 + (z + (x - y)\epsilon)^2}}. \quad (23)$$

We concentrate on the case that the dominant contribution to the second heaviest state is due to the exchange of the right handed neutrino with mass m_2 which couples to the second row in the Dirac mass matrix. The mass eigenstates are given by

$$\nu_3 \simeq \nu_a \quad (24)$$

$$\nu_2 \simeq \frac{|z|e^{i\xi}\nu_e + |r|\nu_b}{\sqrt{z^2 + r^2}} \quad (25)$$

$$\nu_1 \simeq \frac{|r|\nu_e - |z|e^{-i\xi}\nu_b}{\sqrt{z^2 + r^2}}, \quad (26)$$

where

$$\xi = \phi_z - \phi_r$$

$$r = \frac{\sqrt{2}}{z}(zu - a_\nu wy) \quad (27)$$

Note the importance of the higher order terms in determining the lighter neutrino eigenstates. In particular, the $SU(3)$ family symmetry aligns the leading terms in the $(2, 2)$ and the $(2, 3)$ directions and the $(1, 2)$ and the $(1, 3)$ directions; at this order $b'_\nu = c'_\nu = z$, $d'_\nu = a'_\nu = aw$ so the coefficient $a'_\nu b'_\nu - c'_\nu d'_\nu$ vanishes in leading order (this is the second case mentioned above). For this reason the second heaviest neutrino can have a large ν_e component due to the higher

order terms. Let us consider the phenomenological implications of this fit. The mixing angle θ_{23} relevant to atmospheric neutrino mixing is given by

$$\tan^2 \theta_{23} \simeq \left| \frac{z + (x + y)\epsilon}{z + (x - y)\epsilon} \right|^2. \quad (28)$$

Note that it is relatively insensitive to the higher order terms of eq(19).

The mixing angle θ_{12} relevant to solar neutrino mixing is given by

$$\tan^2 \theta_{12} \simeq \left| \frac{z}{r} \right|^2 \quad (29)$$

With these results we may now ask whether it is possible to obtain near bi-maximal mixing with the parameterisation of eq(19), with the parameters constrained to fit the quark masses. As the latter do not determine all the parameters precisely, we will use the remaining freedom to try to generate all the phenomenologically allowed cases, namely the LMA, LOW and VAC solutions although we note that the recent SNO data prefers the LMA solution.

5.2.1 Solution for $\kappa=0$

z	0.91 ± 0.06
x	$-(2.95 \pm 0.1)$
y	0.74 ± 0.07
ϕ_x	0.05 ± 0.01
ϕ_y	0.19 ± 0.01
w	0.57 ± 0.05
u	$-(0.55 \pm 0.12)$
$\bar{\epsilon}$	0.21 ± 0.01
ϵ	0.07 ± 0.01
t_{12}^2	0.58 ± 0.02
t_{23}^2	1.49 ± 0.05
$\left(\frac{M_2}{M_3}\right)^2 <$	$(0.8 \pm 0.4) \times 10^{-5}$
a_u	2.4 ± 0.3

Table 1: The parameters for a fit to the down quark mass matrix under the LOW constraint for θ_{12} in Table 6.

For $\kappa = 0$, i.e. $a_\nu = -3$, the condition assumed above that the right handed neutrinos of mass m_1 and m_2 dominate the see-saw contribution to the heaviest and next heaviest light neutrino masses respectively is $\frac{z^2 \epsilon^6}{m_1} > \frac{9\omega^2 \epsilon^4}{m_2}$. In this case the masses of the light neutrino mass eigenstates are given approximately by

$$M_3 = \frac{\epsilon^6}{m_1} 2z^2 v^2 \quad (30)$$

$$M_2 = \frac{\epsilon^6}{m_2} z^2 (1 + \tan^{-2} \theta_{12}) v^2 \quad (31)$$

$$M_1 < \frac{1}{m_3} v^2 \quad (32)$$

z	0.95 ± 0.01
x	$-(3.15 \pm 0.31)$
y	0.64 ± 0.03
ϕ_x	0.039 ± 0.006
ϕ_y	0.19 ± 0.01
w	0.53 ± 0.03
u	$-(0.5 \pm 0.1)$
$\bar{\epsilon}$	0.22 ± 0.01
ϵ	0.07 ± 0.01
t_{12}^2	1.27 ± 0.04
t_{23}^2	1.4 ± 0.11
$\left(\frac{M_2}{M_3}\right)^2 <$	$(6.0 \pm 1.6) \times 10^{-6}$
a_u	2.7 ± 0.1

Table 2: The parameters for a fit to the down quark mass matrix under the VAC constraint for θ_{12} in Table 6.

Thus for $\frac{z^2 \epsilon^6}{m_1} > \frac{9\omega^2 \epsilon^4}{m_2}$ we have

$$\frac{M_2}{M_3} < \left(\frac{z}{3w}\right)^2 \left(\frac{1 + \tan^{-2} \theta_{12}}{2}\right) \epsilon^2 \quad (33)$$

As we have discussed a large value for θ_{12} appears naturally. However M_2/M_3 is of $O(\epsilon^2)$ and for choices of the parameters z , ω of $O(1)$ and θ_{12} in the observed range, we can not obtain a large enough value for the ratio M_2/M_3 to fit the LMA solution. Thus in this case we can only obtain the LOW and VAC solutions of the data analysis of neutrino oscillations and mixings. The remaining parameters are constrained by the requirement that the parameterisation of equation (19) fits the down quark mass matrix (the first fit of eq(4)). The results are presented in Tables 1 and 2. These may be compared to the fits to neutrino masses and mixing angles coming from the analysis of neutrino oscillations; for convenience we summarise the present situation in the Appendix in Tables 6, 7, 9 and 10. As may be seen we can obtain a good description of both the LOW and VAC solutions.

An important cross check of the structure of the GUT relations used in this analysis comes from the up quark mass matrix. It is given by eq(19) with $\varepsilon = \epsilon$ and $a = a_u = 1$ (since $\kappa = 0$). The prediction $a_u = 1$ is the analogue of the prediction $b' = b$ in eq(9). Given the parameters z and ω from the fit of Tables 1 and 2, we can test this prediction by using the up quark masses to determine a_u . The results are also given in the Tables. We may see that the inclusion of the higher order corrections has actually made the fit to the up quark mass matrix worse than the case ($b' = b$) tested in eq(9) casting doubt on the viability of the $\kappa = 0$ solution.

5.2.2 Solution for $\kappa=2$

For $\kappa = 2$, i.e. $a_\nu = 0$, the condition that the right handed neutrinos of mass m_1 and m_2 respectively dominate the see-saw contribution to the heaviest and next heaviest light neutrino eigenstates masses is $\frac{\epsilon^6}{m_1} > O(\frac{\epsilon^6}{m_2})$. In this case the masses of neutrino mass eigenstates are given

z	0.78 ± 0.14
x	$-(1.86 \pm 0.51)$
y	1.37 ± 0.21
ϕ_x	0.14 ± 0.04
ϕ_y	0.19 ± 0.05
w	0.77 ± 0.06
u	$-(0.87 \pm 0.15)$
$\bar{\epsilon}$	0.18 ± 0.02
ϵ	0.06 ± 0.01
t_{12}^2	0.4 ± 0.1
t_{23}^2	1.7 ± 0.2
$\left(\frac{M_2}{M_3}\right)^2$	$(3 \pm 0.11) \left(\frac{m_1}{m_2}\right)^2$
a_u	1.6 ± 0.4

Table 3: The parameters for a fit to the down quark mass matrix under the LMA constraint for θ_{12} in Table 6.

z	0.95 ± 0.05
x	-2.80 ± 0.13
y	1.37 ± 0.16
ϕ_x	0.09 ± 0.01
ϕ_y	0.18 ± 0.01
w	0.77 ± 0.02
u	-0.86 ± 0.06
t_{12}^2	0.58 ± 0.04
t_{23}^2	1.75 ± 0.18
$\bar{\epsilon}$	0.19 ± 0.03
ϵ	0.07 ± 0.01
$\left(\frac{M_2}{M_3}\right)^2$	$(1.77 \pm 0.11) \left(\frac{m_1}{m_2}\right)^2$
a_u	1.9 ± 0.1

Table 4: The parameters for a fit to the down quark mass matrix for the solution $k = 2$, under the LOW constraint for θ_{12} .

z	0.96 ± 0.06
x	-3.20 ± 0.13
y	0.87 ± 0.13
ϕ_x	0.05 ± 0.01
ϕ_y	0.19 ± 0.01
w	0.61 ± 0.02
u	-0.61 ± 0.04
t_{12}^2	1.23 ± 0.01
t_{23}^2	1.55 ± 0.20
$\bar{\epsilon}$	0.20 ± 0.03
ϵ	0.07 ± 0.01
$\left(\frac{M_2}{M_3}\right)^2$	$(0.82 \pm 0.1) \left(\frac{m_1}{m_2}\right)^2$
a_u	2.4 ± 0.2

Table 5: The parameters for a fit to the down quark mass matrix for the solution $k = 2$ and under the VAC constraint for θ_{12} .

approximately by

$$M_3 = \frac{\epsilon^6}{m_1} 2z^2 v^2 \quad (34)$$

$$M_2 = \frac{\epsilon^6}{m_2} (2u^2 + z^2) v^2 \quad (35)$$

$$M_1 < \frac{1}{m_3} v^2 \quad (36)$$

$$\frac{M_2}{M_3} \approx \frac{2u^2 + z^2}{2z^2} \left(\frac{m_1}{m_2} \right). \quad (37)$$

Since the condition $\frac{z^2 \epsilon^6}{m_1} > O(\frac{\epsilon^6}{m_2})$ is readily satisfied for $m_1/m_2 \lesssim O(1)$ the LMA solution for neutrinos can readily be reproduced through a choice of the ratio m_1/m_2 . The results of the fit to the down quark mass matrix is given in Table 3. This solution is obtained when $\tan \theta_{12}^2 = 0.4$, which is the favoured value for the LMA solution. As may be seen from a comparison with Tables 6, 7 and 8, the agreement with neutrino mixing angles is good.

For the case $\kappa = 2$ the prediction for the up quark mass matrix is that $a_u = 2$. We may see from Table 3 that this is in better agreement with the up quark masses than the case $\kappa = 0$ and also with the case tested in tested in eq(9) when the effects of higher order operators were not included. This is an encouraging indication that the case $\kappa = 2$ may be realised.

It is also possible to obtain the LOW and VAC solutions for $\kappa = 2$. The results are presented in Tables 4 and 5. These may be compared to the fits to neutrino masses and mixing angles coming from the analysis of neutrino oscillations in good agreement with the fits to experiment which are summarised in the Appendix in Tables 6, 7, 9 and 10. In contrast to the $\kappa = 0$ case, the value for a_u is in excellent agreement with the measured up quark masses.

5.3 The prediction for V_{13}

Our analyses has so far investigated the implications for V_{23} and V_{12} which follow from an enhanced flavour and family symmetry. What are the implications for V_{13} ? This is a particularly interesting question for only if V_{13} is quite large will there be any prospect of seeing CP violation in future long baseline neutrino experiments. Due to the form of the hierarchy in the mass matrices, both the charged leptonic mass matrix and the effective neutrino mass matrix, it is possible to diagonalise them by the rotations $R_{23}R_{13}R_{12}$ together with diagonal matrices carrying the phases. From this the 13 ($e3$) V^{MNS} mixing matrix is given by [?, ?]

$$V_{13}^{MNS} = s_{13}^\nu c_{12}^l c_{13}^l e^{i\omega_1} - s_{13}^l c_{12}^l c_{13}^\nu c_{23}^l e^{i\omega_2} + s_{12}^l s_{23}^{L'} c_{13}^\nu e^{i\omega_3} \quad (38)$$

where s_{ij}^f , for $i, j = 1, 2, 3$ and $f = l, \nu$ represent the sines of the mixing angles, analogously for the cosines and the phases ω_i , $i = 1, 2, 3$ are functions of the phases appearing in the mass matrix. The angle $s_{23}^{L'}$ is given by $|c_{23}^\nu s_{23}^l - s_{23}^\nu c_{23}^l e^{i\omega_4}|$. Since the neutrino oscillations experiments are not sensitive to Majorana CP phases, the standard parametrisation used for the case of quarks [?] is commonly used for the case of leptons. In this parameterisation $V_{13} = s_{13} e^{-i\delta}$, where s_{13} should be identified with the sine of the CHOOZ angle and δ is the analogue to the quark CP violation phase. Thus identifying s_{13} with the absolute value of eq(38) we can trace the contributions from charged leptons and neutrinos to the CHOOZ angle.

We can see that, in both the $\kappa = 0$ and $\kappa = 2$ cases discussed above, the value of s_{13}^ν is negligible because of the hierarchy of the Majorana masses we have taken ($m_3 \gg m_2, m_1$). The mixings in the leptonic sector are small and are given by

$$s_{12}^l \approx \frac{|Y_{12}^l - \frac{Y_{13}^l Y_{23}^l}{Y_{33}^l}|}{|Y_{22}^l - \frac{(Y_{23}^l)^2}{Y_{33}^l}|}; \quad s_{13}^l \approx \frac{|Y_{13}^l + \frac{Y_{12}^l Y_{23}^l}{Y_{33}^l}|}{|Y_{33}^l|}, \quad (39)$$

where Y_{12}^l are the elements of the charged lepton mass matrix. From eq(19) we see that $s_{13}^l = O(\bar{\epsilon}^3)$ and $s_{12}^l = O(\bar{\epsilon}^2)$, therefore the latter is the dominant contribution. This result is interesting because $Y_{12}^l/Y_{22}^l \approx \sqrt{m_e/m_\mu}$ and hence from eqs(38) and (39)

$$V_{13}^{MNS} \simeq \sqrt{\frac{m_e}{m_\mu}} \quad (40)$$

which is a testable prediction. This gives a value $V_{13}^{MNS} \approx 0.07$, close to the present bound and large enough for the future CP violating experiments to be viable. This point has been made explicitly by King [?] and the value given is consistent with the results of [?][?]. When the third term in equation (38) is dominant, δ , the analogue to the quark CP violation phase, is $-\omega_3$, which in terms of the elements, $|Y_{ij}^\nu| e^{i\phi_{ij}}$, of the effective neutrino mass matrix it is given by:

$$\delta = -\omega_3 \approx \frac{\gamma_{23}^\nu - \gamma_{12}^\nu - \gamma_{13}^\nu}{2} \quad (41)$$

where

$$\tan \gamma_{ij}^\nu \approx \frac{Y_{ij}^\nu Y_{jj}^\nu \sin(\phi_{ij} - \phi_{jj}) + Y_{ik}^\nu Y_{jk}^\nu \sin(\phi_{ik} - \phi_{jk})}{Y_{ij}^\nu Y_{jj}^\nu \cos(\phi_{ij} - \phi_{jj}) + Y_{ik}^\nu Y_{jk}^\nu \cos(\phi_{ik} - \phi_{jk})} \quad (42)$$

$$\gamma_{23}^\nu \approx \phi_{23} - \phi_{33} \quad (43)$$

for $ij = 12, 13$ and $k \neq i, j$. In the same way as discussed for the quarks, Section 2, we may choose the phases to be ϕ^l , the phase of the $(1, 2)$ element of the Dirac M^ν and χ^l the phase of the $(1, 3)$ element of M^l , then $\delta \approx \phi^l$.

The process of extracting the angles and phases from successive rotations is general and can be applied directly to hierarchical mass matrices; for a further work on this applied to leptons and the implications for CP violation see [?].

5.3.1 $\mu \rightarrow e\gamma$

Several groups [?, ?] have pointed out that off diagonal lepton Yukawa couplings can lead to unacceptably large contributions for lepton family number violation processes such as $\mu \rightarrow e\gamma$. In a supersymmetric framework the branching ratio, $BR(\mu \rightarrow e\gamma)$ has the form

$$BR(\mu \rightarrow e\gamma) \propto |(Y_\nu^\dagger \ln \frac{M_X}{M_R} Y_\nu)_{21}|^2 \tan^2 \beta, \quad (44)$$

where Y_ν is the matrix of Yukawa couplings for Dirac neutrinos (in the basis in which charged leptons are flavour diagonal), M_X is the scale of Grand Unification, M_R is the scale where right handed neutrinos decouple and $\tan \beta$ is equal to the ratio of the vacuum expectation values of the two Higgs doublets of the MSSM. Since $BR(\mu \rightarrow e\gamma)$ depends as well on the soft supersymmetric mass spectrum is useful to determine instead the matrix element $C_{\mu e} = (Y_\nu^\dagger \ln(M_X/M_R) Y_\nu)_{21}$ and hence study the predictions of the structure of Y_ν and the scale of M_R for lepton flavour violating processes [?].

We can compute the element $C_{\mu e}$ for our texture of Y_ν . For the preferred case $\kappa = 2$ we have $C_{\mu e} \approx 6 \times 10^{-3}$ for the LMA solution, $C_{\mu e} \approx 5.6 \times 10^{-3}$ for the LOW solution and $C_{\mu e} \approx 4.2 \times 10^{-3}$ for VAC solution. Even for the case that $\tan \beta$ is large ($h_t \simeq h_b$), due to the hierarchical form of the Yukawa couplings, eq(19), $|(Y_\nu^\dagger Y_\nu)_{21}| \tan \beta$ is small and therefore does not conflict the bounds for $\mu \rightarrow e\gamma$ ⁷. As can be seen from Figure 2 [?] the coefficients $C_{\mu e}$ given above fall below current experimental bounds [?] for a wide range of the soft supersymmetric breaking parameter space ⁸.

5.4 Summary and conclusions

Due to the see-saw mechanism the significant differences between quark and lepton mixing can be explained while keeping the form of their Dirac mass matrices the same. As a result the observed masses and mixings can be accommodated in a theory in which there is a very large underlying family and flavour symmetry group. We have explored the phenomenological implications of an $SU(3)$ family symmetry together with a $SO(10)$ GUT flavour symmetry and additional Abelian family symmetries, chosen to restrict the allowed Yukawa couplings. Allowing for spontaneous (perturbative) breaking of this group we found a symmetry breaking scheme in which the observed hierarchical quark masses and mixings are quantitatively described together

⁷The structure of the effective mass matrix for low energy neutrinos (provided by the see-saw mechanism) in the models considered here belongs to the Class 3 of reference [?].

⁸To compare with the bounds presented in [?] we need to multiply the upper bounds for $C_{\mu e}$ by 10/50, since $\tan \beta \approx 50$.

with the hierarchy of charged lepton masses and an hierarchical structure for neutrino masses with near bi-maximal mixing in the lepton sector. In this the presently unknown mixing angle, $(\theta_{MNS})_{13}$, is determined mainly by the mixing in the lepton sector. While smaller than the other mixing angles, it is close to the present limits, and is large enough to allow for significant CP violating effects to be visible in future long baseline neutrino experiments.

Given the very large underlying symmetry, the fermion masses are heavily constrained. The perturbative breaking ensures an hierarchical structure for the masses and in terms of the breaking parameters, the order of magnitude of the ratios of the quark and lepton masses is determined once the family symmetry properties of the fields are determined. For quarks, having constrained the family charges to fix the down quark mass ratios, one obtains one order of magnitude prediction for the ratio of up quark masses which is in good agreement with experiment. Further the family symmetry ensures an approximate texture zero in the $(1,1)$ matrix element which gives rise to the successful GST relation between the CKM mixing and the mass ratios of the first two generations of up and down quarks.

The extension of the family symmetry to leptons can be done together with via an enlarged GUT flavour symmetry, $SO(10)$. This results in predictions for the charged lepton masses in terms of the symmetry breaking pattern of $SO(10)$. For one particular choice, the predictions are in good agreement with measurement, reproducing the Georgi Jarlskog structure for the light lepton masses. In addition the $SO(10)$ symmetry relates the Yukawa couplings of the up and down quark sectors and replaces the order of magnitude prediction for the ratio of up quarks by an absolute prediction. For the preferred choice of symmetry breaking this prediction is in excellent agreement with experiment.

The neutrino mass matrices are also strongly constrained by the symmetry. If the expansion parameter for the Majorana masses is smaller than that for the up quarks, the Majorana mass matrix is approximately diagonal in the family symmetry basis. In this case the lepton mixing angles are determined by the Dirac mass matrices for the charged leptons and neutrinos and these in turn are related by the GUT symmetry to the Dirac mass matrices of the quarks. As a result the lepton mixing angles⁹ are determined by the properties of the quarks. The mixing angle relevant to atmospheric neutrino mixing is well determined by the leading order operators in the symmetry breaking expansion. Through vacuum alignment in the $SU(3)$ family sector this mixing angle is large and consistent with the measure atmospheric mixing angle. The solar mixing angle turns out to be quite sensitive to subdominant terms in the symmetry breaking expansion. As a result one obtains only an order of magnitude prediction for this angle determining it to be of $O(1)$. For the favoured $SO(10)$ symmetry breaking pattern the coefficients of the sub-dominant operators needed to obtain a quantitative agreement of this mixing angle with data are consistent with the constraints on the expansion which follow from fitting the down quark mass matrix structure. Finally the value of $(\theta_{MNS})_{13}$ is determined to be approximately $\sqrt{m_e/m_\mu}$, coming mainly from the charged lepton sector.

To summarise, the data on all quark and lepton masses and mixings is qualitatively consistent with a significantly enlarged family and flavour symmetry. Given the disparity between these quantities in the quark and lepton sectors this is already remarkable. On a more quantitative level, a specific pattern of family symmetries and symmetry breaking leads to a quantitative prediction for seven of the parameters of the Standard Model, namely $(\theta_{CKM})_{12}$, $m_u m_t / m_c^2$,

⁹The neutrino masses, while constrained in some case, are largely determined by the unknown Majorana mass eigenvalues.

m_e, m_μ, m_τ and $(\theta_{MNS})_{23}$ together with an order of magnitude prediction for $(\theta_{MNS})_{13}$. Such agreement is encouraging and suggests one should try to construct a complete underlying theory, be it a SUSY GUT or perhaps a superstring theory.

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A Appendix

	LMA	LOW	VAC	Ref.
$\tan^2 \theta_{12}$	$(4.0 \pm 1) \times 10^{-1}$	$(5.8 \pm 4) \times 10^{-1}$	(1.23 ± 0.1)	
$\frac{ M_2 ^2}{ M_3 ^2}$	$3 \frac{m_1^2}{m_2^2}$	$1.77 \frac{m_1^2}{m_2^2}$	$8.2 \times 10^{-1} \frac{m_1^2}{m_2^2}$	
$(\tan^2 \theta_{12})_{\text{exp}}$	4.0×10^{-1}	5.8×10^{-1}	$(0.68, 1.8)$	[?, ?, ?]
$\frac{ \Delta m_{12}^2 }{ \Delta m_{23}^2 _{\text{exp}}}$	$(0.3, 33.6) \times 10^{-2}$	$(0.48, 10) \times 10^{-5}$	$\approx 1.16 \times 10^{-7}$	[?]
$\frac{ \Delta m_{12}^2 }{ \Delta m_{23}^2 _{\text{exp}}}$	$(0.5, 30.8) \times 10^{-2}$	$(0.73, 10) \times 10^{-5}$	$\approx 1.44 \times 10^{-7}$	[?]

Table 6: Comparison between the predictions for $k = 2$ and the experiments for $\tan^2 \theta_{12}$ and $|\frac{M_2}{M_3}|^2$ for the LMA, LOW and VAC solutions for solar neutrinos. The confidence levels presented are at 3σ .

Atmospheric+chooz information						
Δm_{32}^2		t_{23}^2		t_{13}^2		Ref.
BFP	Range	BFP	Range	BFP	at 90 %CL	
3.1×10^{-3}	$(1.1, 7.3) \times 10^{-3}$	1.4	$(0.33, 3.7)$	0.005	< 0.055	[?]
	at 99 %CL		at 99 %CL			
2.5×10^{-3}	$(1.2, 4.8) \times 10^{-3}$	1	$(0.43, 2.3)$			[?]
	at 3σ		at 3σ			

Table 7: The constraints on neutrino mixing parameters coming from Atmospheric neutrino data and from CHOOZ. The second analysis cited has been performed with updated atmospheric data. For different values of t_{13}^2 there are different two dimensional C.L. regions for the variables t_{23} and Δm_{23}^2 ; here we present the appropriate for the value of t_{13}^2 obtained with the mass matrix structure discussed here.

LMA, solar information					
Δm_{12}^2 (eV) ²		t_{12}^2		g.o.f.	Ref.
BFP	at 3 σ CL	BFP	at 3 σ CL		
5.5×10^{-5}	$(0.23, 3.7) \times 10^{-4}$	4.2×10^{-1}	$(2.4, 8.9) \times 10^{-1}$	49%	[?]
5.6×10^{-5}	$(0.22, 2.2) \times 10^{-4}$	3.9×10^{-1}	$(2.0, 6.4) \times 10^{-1}$		[?]
6.2×10^{-5}	$(0.23, 3.4) \times 10^{-5}$	4.0×10^{-1}	$(2.3, 7.9) \times 10^{-1}$	84%	[?]
	at 1 σ CL		at 1 σ CL		
4.5×10^{-5}	$(3.1, 7.2) \times 10^{-1}$	4.0×10^{-1}	$(3.2, 4.8) \times 10^{-1}$		[?]

Table 8: Experimental constraints for LMA solution. BFP refers to the best fit point given in the cited references, the 3 σ =99.73% CL or the 1 σ C.L. is shown.

LOW, solar information					
Δm_{12}^2	Δm_{12}^2	t_{12}^2	t_{12}^2	g.o.f.	Ref.
BFP	at 3 σ CL	BFP	at 3 σ CL		
1.0×10^{-7}	$(0.5, 1.1) \times 10^{-7}$	7.1×10^{-1}	$(5.5, 10) \times 10^{-1}$	45%	[?]
1.1×10^{-7}	$(0.6, 1.2) \times 10^{-7}$	6.9×10^{-1}	$(4, 9) \times 10^{-1}$	69%	[?]
	$(0.9, 1) \times 10^{-7}$		$(6, 7.5) \times 10^{-1}$		[?]

Table 9: Experimental constraints for the LOW solution. BFP refers to the best fit point given in the cited references, the 3 σ CL has been estimated from those references. The analyses cited have been performed using the latest SNO information.

VAC					
Δm_{12}^2	Δm_{12}^2	t_{12}^2	t_{12}^2	g.o.f.	Ref.
BFP	at 3 σ CL	BFP	at 3 σ CL		
4.6×10^{-10}	$(3.5, 5.7) \times 10^{-10}$	2.4×10^0	$(0.3, 3.5)$	42%	[?]

Table 10: Experimental constraints for VAC solution. BFP refers to the best fit point given in the cited references, the 3 σ CL has been estimated from those references. The analysis cited has been performed using the latest SNO information and with an enhanced CC cross section for deuterium, as quoted in [?].